stress-strain relation (1) and subsequently differentiated with respect to the space coordinates, to yield

$$\sigma_{ij,j}{}^{d} * dJ = -\rho \sum_{n} B_n \omega_n{}^2 X_{in}(\mathbf{x}) \varphi_n(t) J_R \tag{21}$$

Taking the Stieltjes convolution of the equation of motion (13) with $J(t)$, we obtain

$$\sigma_{ij,j}{}^{d} * dJ = \rho \ddot{u}_i{}^{d} * dJ - (\rho/J_R) U_i{}^{s}(\mathbf{x}) \ddot{J}(t) * dJ \tag{22}$$

By substituting (17) and (18) in (22), the latter equation is rewritten as

$$\sigma_{ij,j} *^{d} dJ = \rho \sum_{n} B_n X_{in}(\mathbf{x}) \; [\ddot{\varphi} * dJ - (1/J_R) J * dJ] \tag{23}$$

By comparing (21) and (23) we conclude that

$$\ddot{\varphi}_n(t) * dJ = -\omega_n{}^2 \varphi_n(t) J_R + (1/J_R) \ddot{J} * dJ \tag{24}$$

Equation (24) can be solved by means of the Laplace transform technique.

Returning to Eq. (7), we find by employing (11) and (18) that the solution of the viscoelastic vibration problem is of the form

$$u_i(\mathbf{x}, t) = (1/J_R) \; U_i{}^{s}(\mathbf{x}) J(t) - \sum_{n} B_n X_{in}(\mathbf{x}) \varphi_n(t) \tag{25}$$

It is concluded that for a body of viscoelastic material of constant Poisson's ratio subjected to surface tractions, the dynamic displacement can be obtained, provided that 1) the static boundary-value problem for the corresponding elastic body subjected to the surface tractions can be solved, 2) the normal modes of free vibration of the corresponding elastic body are known, and 3) Eq. (24) can be solved for $\varphi_n(t)$.

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Comment on "A Mathematical Model of the Cyclic Stress-Strain Relationships"

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THE interest in damping from the point of view of engineering is two fold; first is the limitation of resonant vibration by energy dissipation within the component and second is the consequence of this energy dissipation, the fatigue of metals. The stress-strain relationship under cyclic load manifests itself in the form of a hysteresis loop, the area of which is proportional to the energy dissipated per cycle.

Dr. Knackstedt[12] has developed an analytical model based on the nonlinearity of the stress-strain curve which makes it possible to determine certain quantities in connection with the damping properties of a metal. He also expresses his intention of using these mathematical expressions to assess the validity of Poschl's hypothesis of fatigue based on an energy criterion. It is in this respect that this paper is of great interest to us as we have made an analytical study of fatigue for lives in excess of 10^5 cycles using a microplastic strain energy accumulation criterion.[1]

Under cyclic loading, a large amount of energy is dissipated by the vibrating component. A comparison of the total energy input with the maximum storable energy shows the existence of different energy dissipating mechanisms, all of which are not necessarily damaging in the fatigue sense. Damping can be observed at stress levels below the fatigue limit of the metals and arises from effects that are related to aspects of the crystal structure and other physical properties. Therefore, an approach based on the stress-strain curve alone causes a large loss of generality because of the fact that the mechanisms of damping[2–4] are much more involved than can be deduced from an analysis of tensile behavior alone. In this respect, it is relevant to point out that it is not possible to use the total area of the hysteresis loop as an index of fatigue damage in the long-life region for the following reasons.

It is more appropriate to divide the origins of damping into two sections, the first dealing with the mechanisms at low strains and the second with the contributions at strain levels above a "critical" value.[3] The marked distinction between these two aspects is due to the fact that the latter type of damping is considered to result from plastic slip, that is, irreversible dislocation movement which is also necessary for fatigue failure. Since these two "nondamaging" and "damaging" types of energy tranformation occur simultaneously, and until an advanced degree of plasticity overlap,[10] no method as yet has been found to delineate the plastic portion from the observed hysteresis loops.

In mechanical testing, the elastic limit is measured as a bulk property and is useful so as to provide a common basis for design. However, the prediction of mechanical behavior under complex service conditions requires an additional and more detailed analysis than the conventional practice of stress analysis. In some problems such as creep and fatigue, where the failure is initiated at a microscopic level, the elastic limit is insufficient as an indication of the onset of plasticity. It is, therefore, necessary to consider not only the applied load but also the detailed mechanical properties and even the microstructure of the material. This necessity arises because of the fact that, at a microstructural level, all metals are inhomogeneous and anisotropic, so that the observed bulk properties represent a statistical average of these microscopic properties.

The microstructural characteristics of metals give rise to microinhomogeneity of stresses and strains in a metal under stress and also lead to the onset of plasticity in the form of isolated "microplastic" elements in a metal at stresses below the nominal elastic limit.[5] Above a certain (critical) value of stress which is below the elastic limit, a very small portion of an observed hysteresis loop is of plastic strain energy. As the stress is raised, the number of plastic elements increases and at an advanced degree of plasticity, the area of the hysteresis loop can be considered to be representative of the plastic strain energy dissipated per cycle. Therefore, at high stresses the area of the hysteresis loop can be used as an index of fatigue damage in order to make an analytical study of fatigue. Stowell[11] has reported a successful application of this concept in the low cycle region. However, in this region the differences between the monstonic and the cyclic stress-strain curve are not negligible and difficulties may be experienced because of cyclic hardening and softening.

The application of this concept in the long-life region presents certain difficulties due to the fact that though the material is nominally elastic, at a microscopic level an elastoplastic condition exists.[5] The mathematical expressions

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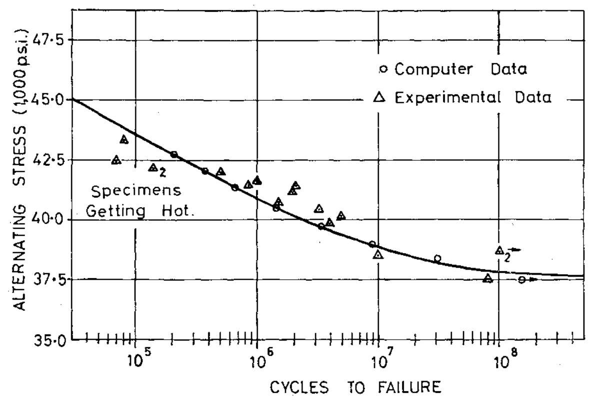

Fig. 1 Comparison of computer results with experimental data.

based on the classical strength theories are not adequate to define this physical condition. Feltner and Morrow[10] have assumed that a logarithmic plot of static true stress vs true plastic strain is valid when extrapolated back into the elastic stress region and have calculated the plastic energy dissipated per cycle using this empirical relationship. Such extrapolation techniques are not general enough to cover the microscopic elastic-plastic cases as the problem is of a statistical nature. The occurrence of plasticity in the form of isolated pockets and the extent of inelastic deformation of plastic elements are probabilistic events dependent upon the microstructure of the material. This requires a statistical approach to the strength of materials and certain microstructure sensitive experimental techniques[6,7] to define the value of the critical stress (the true elastic limit) and, hence, the statistical functions.

The dispersion of the values of stresses, strains, and elastic limits of the microelements were considered by a statistical analysis[5] and by employing the experimental technique in Ref. 6, a mathematical model for generating microplastic hysteresis loops was constructed.[8] In this analysis, the plastic strain energy dissipated per cycle was determined as a "probabilistic" quantity which cannot be arrived at by formulations based on the bulk properties of material alone. The fatigue failure was assumed to occur when this energy accumulated to a threshold value—equal to the area under the true stress—true strain diagram.[10,11]

The analysis of the plastic strain energy absorbed per cycle and the summing of this energy to fracture was carried out by computer and the points obtained compared extremely well with the experimental points. This method of analysis was successfully applied to six types of steels,[1,9] two of which have already been reported in Ref. 1 (and one is shown in Fig. 1).

Dr. Knackstedt's suggestion that the fatigue life is not connected with the total amount of energy expended during fatigue testing is very true and as our results indicate, a plastic energy approach is a very successful form of analysis provided that the mathematical expressions are constructed so as to represent the actual physics of the phenomenon at a microstructural level. Furthermore, the correlation between the true energy to static fracture and the total energy to fatigue fracture indicates a connection between the two which is inexplicable at present. Though this assumption does not explain the mechanisms of fatigue, nevertheless, both forms of fracture represent a certain amount of work that has to be done to sever the atomic bonds of the metal where the means are different but the end results are the same. We are sure that it will be a valuable contribution if Dr. Knackstedt can establish the validity of this assumption.

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Reply by Author to Comments by A. Esin and W. J. D. Jones

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The Comment by A. Esin and W. J. D. Jones[1] is well taken, as far as they point out that and how the mechanisms of damping are much more involved than can be deduced from an analysis of tensile behavior alone. However, I did not seek to verify a failure mechanism, but rather a bulk material property.

As may be derived from the Introduction of the subject paper,[2] the mathematical model of the cyclic stress-strain relationship was established for the purpose of arriving at an analytical expression which could be used to derive numerical values of the total mechanical work that would be transferred to the specimen. I think there can be no doubt that the area enveloped by the measured hysteresis loop, representing the cyclic engineering stress-engineering strain relationship, determines this work per unit of the initial volume and cycle. By comparing the hysteresis loop with the equivalent ellipse, the order of magnitude of the exponents m or n of the plastic term in strain- or stress-controlled cyclic tests could be derived.

The measured heat produced in the specimen from the mechanical work transferred to the specimen must be subtracted in order to find the work that in Pöschl's assumption is responsible for the fatigue failure. But, still at high stresses the area of the hysteresis loop cannot then be used as an index of fatigue damage.

Note: On p. 1824 of the original article, the last equation in Sec. 222 should read:

$$\sigma_0 = \tfrac{1}{3}[2\sigma_0' + (d\sigma_0'/d\epsilon_0)\,\epsilon_0]$$

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